Comparison of various approaches to portfolio efficiency

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Abstract. This paper deals with portfolio efficiency testing with respect to various criteria. Basically, two approaches can be employed. If expected utility approach is considered one can test portfolio efficiency with respect to stochastic dominance relation. We focus on the second-order stochastic dominance portfolio efficiency that allows for risk averse decision makers. Alternatively, portfolio efficiency with respect to mean-risk criteria is analyzed, when considering the most favorite risk measures (variance, semivariance, value at risk, conditional value at risk). We assume discrete distribution of monthly returns. As the basic assets, we consider ten representative US industry portfolios and a riskfree asset. For all these efficiency approaches we test more than forty thousand portfolios from a regular grid and we identify sets of efficient portfolios. We compare these sets and corresponding efficient frontiers between each other in classical mean-variance space.

Keywords: mean-risk models, risk measures, second-order stochastic dominance, portfolio efficiency.

JEL classification: D81, G11
AMS classification: 91B16, 91B30

1 Introduction

The classical portfolio efficiency analysis is based on well-known mean-variance criteria introduced already in 1952 by Harry Markowitz (see [10]). Since that time, many improvements have been proposed and implemented. The new risk measures, such as semivariance [11], Value at Risk [4] (VaR) or Conditional Value at Risk [13] (CVaR) where proposed and analysed. These alternative measures model the risk of investments in a more sophisticated way, focusing more on the investments losses.

Stochastic dominance (SD) is another appealing approach to analyzing investments and portfolio choice problems. Stochastic dominance relations offer an approach that effectively considers the entire return distribution rather than a finite set of moments. Assuming risk averse investors, we limit our attention to second-order stochastic dominance relation. We say that portfolio $\boldsymbol{\lambda}$ dominates portfolio $\boldsymbol{\tau}$ with respect to second-order stochastic dominance if expected utility of $\boldsymbol{\lambda}$ is not lower than expected utility of $\boldsymbol{\tau}$ for all concave utility functions. Put differently, if portfolio $\boldsymbol{\lambda}$ dominates portfolio $\boldsymbol{\tau}$ with respect to second-order stochastic dominance then no risk averse investor prefers $\boldsymbol{\tau}$ to $\boldsymbol{\lambda}$. The more general notion of first-order portfolio efficiency was discussed in [8] and [7].

In the last decade, several portfolio efficiency tests with respect to second-order stochastic dominance were developed. First test, based on the representative set of utility functions, was developed in 2003 [12]. A year later, another test using majorization theorem for dual stochastic dominance approach were introduced, see [8]. Finally, [6] presents a test formulated in terms of CVaRs.

The aim of this paper is to analyze the efficiency of portfolios with respect to mean-risk criteria and empirically compare it with SSD portfolio efficiency. We consider 2001 - 2010 period of monthly returns of ten representative US industry portfolios and a riskfree asset. For this data set, we construct more than 40 000 portfolios from a regular grid. For each portfolio, we test its efficiency with respect to mean-risk criteria and with respect to SSD criterion. Finally, we construct and compare the empirical efficient frontiers and sets of efficient portfolios.

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The remainder of this paper is structured as follows. Section 2 introduces basic definitions of second-order stochastic dominance relation and portfolio efficiency with respect to this criterion. Section 3 presents mean-risk efficiency algorithms (tests) in terms of mathematical programming, when variance, semivariance, VaR and CVaR are employed. It is followed by an empirical study where we compare the corresponding efficient sets and frontiers. Section 5 summarizes and concludes the paper.

2 Portfolio efficiency with respect to second-order stochastic dominance relation

We consider a random vector $\mathbf{r} = (r_1, r_2, \dots, r_N)$ of returns of N assets with a discrete probability distribution described by T equiprobable scenarios. The returns of the assets for the various scenarios are given by

$$X = \begin{pmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \\ \vdots \\ \mathbf{x}^T \end{pmatrix}$$

where $\mathbf{x}^t = (x_1^t, x_2^t, \dots, x_N^t)$ is the t-th row of matrix X representing the assets returns along t-th scenario. We assume that the decision maker may also combine the alternatives into a portfolio. We will use $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_N)^{\mathrm{T}}$ for a vector of portfolio weights and $X\boldsymbol{\lambda}$ represents returns of portfolio $\boldsymbol{\lambda}$. The portfolio possibilities are given by a simplex

$$\Lambda = \{\boldsymbol{\lambda} \in R^N | \mathbf{1}'\boldsymbol{\lambda} = 1, \ \ \lambda_j \geq 0, \ \ j = 1, 2, \dots, N\},$$

which arises as the relevant case if we exclude short sales and impose a budget restriction. Moreover, the tested portfolio is denoted by $\boldsymbol{\tau}$.

Following [9] and references therein, portfolio $\boldsymbol{\lambda}$ *dominates* portfolio $\boldsymbol{\tau}$ *with respect to second-order stochastic dominance* ($\boldsymbol{\lambda} \succ_{SSD} \boldsymbol{\tau}$) if $Eu(\mathbf{r}\boldsymbol{\lambda}) \geq Eu(\mathbf{r}\boldsymbol{\tau})$ for all non-decreasing and concave utility functions with strict inequality for at least one such utility function. Alternatively, one can consider as a definition of this relation some of its necessary and sufficient conditions summarized in, for example, [5]. In any case, if portfolio $\boldsymbol{\lambda}$ dominates portfolio $\boldsymbol{\tau}$ with respect to second-order stochastic dominance then every risk averse decision maker prefers $\boldsymbol{\lambda}$ to $\boldsymbol{\tau}$ or is indifferent between them.

Following [12], [8] and [6], we define the efficiency of a given portfolio with respect to second-order stochastic dominance relative to all portfolios that can be created from a considered set of assets.

Definition 1. A portfolio $\boldsymbol{\tau}$ is *SSD inefficient* if there exists portfolio $\boldsymbol{\lambda} \in \Lambda$ such that $\boldsymbol{\lambda}$ dominates $\boldsymbol{\tau}$ by SSD. Otherwise, the portfolio $\boldsymbol{\tau}$ is *SSD efficient*.

Since its relation to CVaR (one of the considered risk measure), we choose the portfolio efficiency test developed in [6]. Let $\alpha_k = k/T, \ \ k \in K = \{0, 1, \dots, T-1\}$. Consider the following linear program:

$$\begin{aligned} D^*(\boldsymbol{\tau}) = \max_{D_k, \lambda_n, b_k, w_k^t} & \sum_{k=1}^{T} D_k \\ \text{s.t. } \mathrm{CVaR}_{\frac{k-1}{T}}(-\mathbf{r}'\boldsymbol{\tau}) - b_k - \frac{1}{(1-\frac{k-1}{T})T} \sum_{t=1}^{T} w_k^t & \geq D_k, \qquad k \in K \\ w_k^t + \mathbf{x}^t\boldsymbol{\lambda} & \geq -b_k, \ \ t, k \in K \\ w_k^t & \geq 0, \qquad t, k \in K \\ D_k & \geq 0, \qquad k \in K \\ \boldsymbol{\lambda} & \in \Lambda. \end{aligned}$$

The optimal objective value $D^*(\boldsymbol{\tau})$ can be seen as a measure of SSD inefficiency of portfolio $\boldsymbol{\tau}$ and [6] uses it for the portfolio efficiency testing as follows.

Theorem 1. *If $D^*(\boldsymbol{\tau}) > 0$ then $\boldsymbol{\tau}$ is SSD inefficient and $\boldsymbol{\lambda}^* \succ_{SSD} \boldsymbol{\tau}$. Otherwise, $D^*(\boldsymbol{\tau}) = 0$ and $\boldsymbol{\tau}$ is SSD efficient.*

3 Portfolio efficiency with respect to mean-risk criteria

The classical optimization task which leads to mean-risk efficient portfolios can be written as:

$$\begin{aligned} &\min_{\boldsymbol{\lambda}} risk_{\boldsymbol{\lambda}} \\ &\text{s. t. } mean_{\boldsymbol{\lambda}} \geq mean_e \\ &\qquad \boldsymbol{\lambda} \in \Lambda, \end{aligned} \tag{1}$$

where $risk_{\boldsymbol{\lambda}}$ and $mean_{\boldsymbol{\lambda}}$ represent risk and mean return of portfolio $\boldsymbol{\lambda}$, respectively, $mean_e$ is the minimal required expected return.

Definition 2. A portfolio $\boldsymbol{\tau}$ is *mean-risk inefficient* if there exists portfolio $\boldsymbol{\lambda}$ satisfying $risk_{\boldsymbol{\lambda}} \leq risk_{\boldsymbol{\tau}}$ and $mean_{\boldsymbol{\lambda}} \geq mean_{\boldsymbol{\tau}}$ with at least one strict inequality. Otherwise, portfolio $\boldsymbol{\tau}$ is *mean-risk efficient.*

Since we want to test the efficiency of portfolio $\boldsymbol{\tau}$ with respect to mean-risk criterion we reformulate (1) in the following way:

$$\begin{aligned} \xi(\boldsymbol{\tau}) = &\min_{\boldsymbol{\lambda}, s_m, s_r} \; s_m + s_r \\ &\text{s. t. } mean_{\boldsymbol{\lambda}} - mean_{\boldsymbol{\tau}} \geq s_m \\ &\qquad risk_{\boldsymbol{\tau}} - risk_{\boldsymbol{\lambda}} \geq s_r \\ &\qquad \boldsymbol{\lambda} \in \Lambda \\ &\qquad s_m, s_r \geq 0, \end{aligned} \tag{2}$$

where $\xi(\boldsymbol{\tau})$ can be understood as a measure of mean-risk inefficiency that is composed from possible improvements in both mean and risk. These improvements are represented by slack variables s_m and s_r.

Theorem 2. *If $\xi(\boldsymbol{\tau}) > 0$ then $\boldsymbol{\tau}$ is mean-risk inefficient. Otherwise, $\xi(\boldsymbol{\tau}) = 0$ and $\boldsymbol{\tau}$ is mean-risk efficient.*

In all mean-risk efficiency tests, we will use $mean_{\boldsymbol{\lambda}} = \frac{1}{T}\sum_{t=1}^{T} \mathbf{x}^t \boldsymbol{\lambda}$ and $mean_{\boldsymbol{\tau}} = \frac{1}{T}\sum_{t=1}^{T} \mathbf{x}^t \boldsymbol{\tau}$. To get the test for mean-variance efficiency, we apply $r_{\boldsymbol{\lambda}} = \boldsymbol{\lambda}' V \boldsymbol{\lambda}$ and $r_{\boldsymbol{\tau}} = \boldsymbol{\tau}' V \boldsymbol{\tau}$ to (2) where V is the covariance matrix of asset returns. In the semivariance case, we use variables z^t corresponding to returns of portfolio $\boldsymbol{\tau}$ that are smaller than $mean_{\boldsymbol{\tau}}$. The general test (2) is modified as follows:

$$\begin{aligned} \xi(\boldsymbol{\tau}) = &\min_{\boldsymbol{\lambda}, s_m, s_r, z^t} \; s_m + s_r \\ &\text{s. t. } mean_{\boldsymbol{\lambda}} - mean_{\boldsymbol{\tau}} \geq s_m \\ &\qquad semivariance_{\boldsymbol{\tau}} - \frac{1}{T}\sum_{t=1}^{T} \left(z^t\right)^2 \geq s_r \\ &\qquad z^t \geq -\mathbf{x}^t \boldsymbol{\lambda} + mean_{\boldsymbol{\lambda}}, \; t = 1, .., T \\ &\qquad z^t \geq 0, \; t = 1, .., T \\ &\qquad \boldsymbol{\lambda} \in \Lambda \\ &\qquad s_m, s_r \geq 0. \end{aligned}$$

In the case of VaR we have to employ integer variables δ^t, what leads to more computationally demanding problem:

$$\begin{aligned} \xi(\boldsymbol{\tau}) = &\min_{\nu, \boldsymbol{\lambda}, \delta^t, s_m, s_r} \; s_m + s_r \\ &\text{s. t. } mean_{\boldsymbol{\lambda}} - mean_{\boldsymbol{\tau}} \geq s_m \\ &\qquad \mathrm{VaR}_{\boldsymbol{\tau}} - \nu \geq s_r \\ &\qquad -\mathbf{x}^t \boldsymbol{\lambda} \leq \nu + K\delta^t, \; t = 1, .., T \\ &\qquad \sum_{t=1}^{T} \delta^t = \lfloor (1-\alpha) T \rfloor \\ &\qquad \delta^t \in \{0, 1\}, \; t = 1, .., T \\ &\qquad \boldsymbol{\lambda} \in \Lambda \\ &\qquad s_m, s_r \geq 0, \end{aligned}$$

where $\lfloor x \rfloor = \max\{n \in \mathbb{N}_0, n < x\}$ for $x \in \mathbb{R}^+$, and K is sufficiently large constant, for example, $K \geq \max_{t,j} x_j^t - \min_{t,j} x_j^t$. When CVaR is chosen as the risk measure, we get the following linear program:

$$\begin{aligned}
\xi(\boldsymbol{\tau}) = \min_{\boldsymbol{\lambda}, s_m, s_r, z^t, a} \quad & s_m + s_r \\
\text{s. t.} \quad & mean_{\boldsymbol{\lambda}} - mean_{\boldsymbol{\tau}} \geq s_m \\
& \mathrm{CVaR}_{\boldsymbol{\tau}} - a - \frac{1}{(1-\alpha)T} \sum_{t=1}^{T} z^t \geq s_r \\
& z^t \geq -\mathbf{x}^t \boldsymbol{\lambda} - a, \; t = 1, .., T \\
& z^t \geq 0, \; t = 1, .., T \\
& \boldsymbol{\lambda} \in \Lambda \\
& s_m, s_r \geq 0.
\end{aligned}$$

4 Empirical application

We consider monthly returns of ten US representative industry portfolios and a risk free asset which represent $N = 11$ basic assets. The returns can be found in data library of Kenneth French [14] and we proxy risk free asset by CRSP index. We consider ten years period 2001 - 2010, that is $T = 120$ historical scenarios. The Table 1 shows descriptive statistics of the data set. We consider a regular grid

	mean	st. dev.	min	max	skewness
Non-durables	0.622	3.583	-12.980	9.310	-0.622
Durables	0.689	8.515	-32.790	43.090	0.462
Manufactory	0.778	5.525	-20.840	17.950	-0.664
Energy	1.038	5.991	-17.020	19.160	-0.320
HiTech	0.310	7.879	-26.230	19.400	-0.385
Telecom	0.116	5.956	-15.460	21.380	0.027
Shops	0.531	4.773	-15.060	12.310	-0.390
Health	0.097	3.968	-10.930	9.220	-0.391
Utilities	0.519	4.487	-12.480	10.200	-0.924
Other	0.157	5.594	-19.560	16.280	-0.619
Riskfree	0.180	0	0.180	0.180	0

Table 1: The basic descriptive statistics.

on feasibility set Λ with step size $\frac{1}{8}$ and we create 43 758 portfolios from these assets. Specifically, each element of each created portfolio $\boldsymbol{\lambda}$ is equal to one of the following numbers: $0, \frac{1}{8}, \frac{2}{8}, ..., 1$ such that the elements of each portfolio sum up to 1, in order to satisfy the conditions of set Λ.

Firstly, applying Theorem 1, we test SSD efficiency of all considered portfolios. We find only 0,11% SSD efficient portfolios and Figure 1 (left) presents the set of SSD efficient portfolios in the classical mean-variance space.

Secondly, we identify a set of mean-variance, mean-semivariance, mean-VaR and mean-CVaR efficient portfolios using tests in Section 3. Perhaps surprisingly, the set of mean-semivariance efficient portfolios coincides with that for mean-CVaR criteria. For all risk measures we construct efficient frontiers: mean-variance frontier (solid line), mean-semivariance and mean-CVaR frontier (dashed line), mean-VaR frontier (dashdotted line) and we compare the results in Figure 1 (right). These frontiers are only empirical ones, that is, they are constructed only from portfolios on the grid. For large values of mean (larger than 0.88%), the mean-variance frontier coincides with mean-semivariance and mean-CVaR ones.

Finally, we compare the efficiency sets among each other. We find in our study that mean-semivariance efficiency is equivalent to mean-CVaR efficiency. Moreover, each of these efficient portfolios is also efficient with respect to mean-variance criteria. Finally, all mean-risk efficient portfolios, except of mean-VaR efficient ones, are classified as SSD efficient as well. Figure 2 summarizes this efficiency sets comparison.

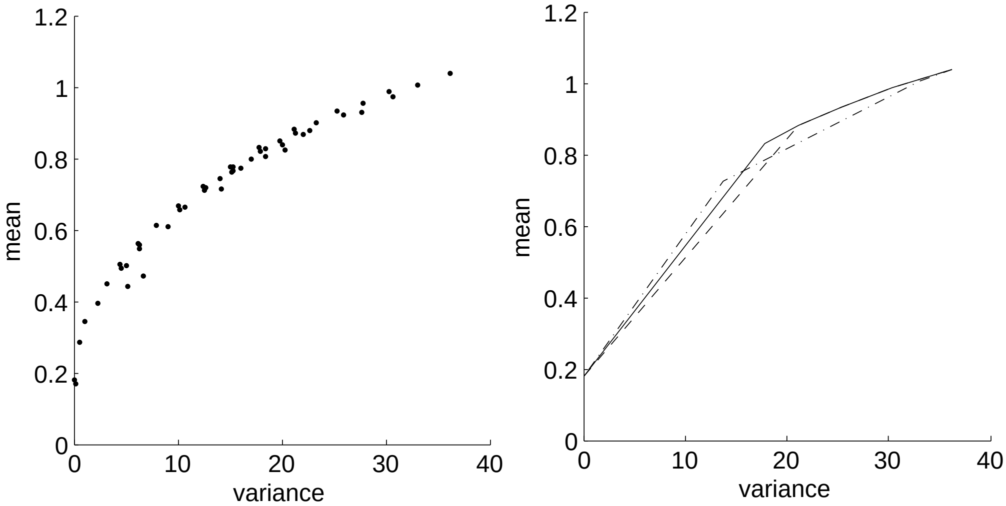

Figure 1: Empirical set of SSD efficient portfolios (left) and mean-risk efficiency frontiers (right).

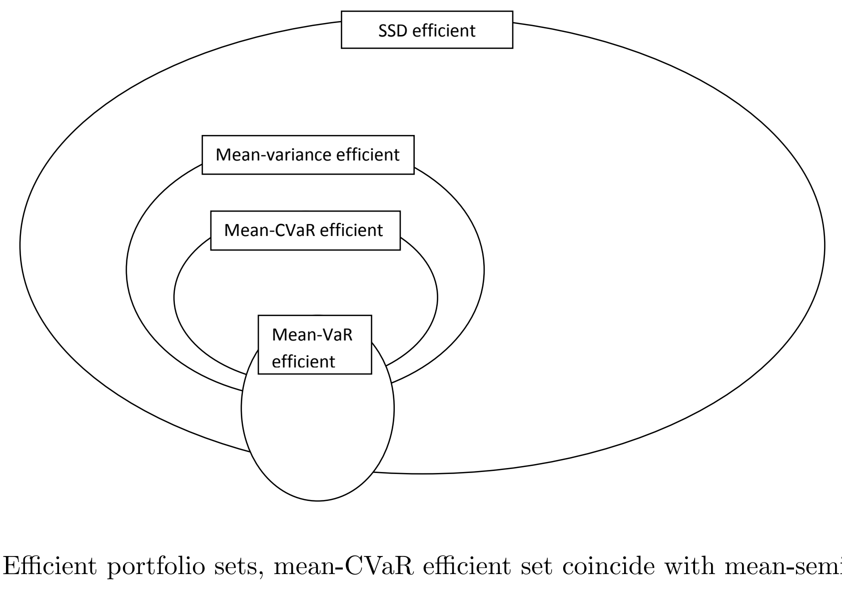

Figure 2: Efficient portfolio sets, mean-CVaR efficient set coincide with mean-semivariance one.

5 Conclusion

This paper compares the several portfolio efficiency sets when using stochastic dominance and mean-risk criteria. We constructed and compared these sets and corresponding efficient frontiers. We found that SSD efficiency set is larger than any mean-risk efficiency set. This is not a surprising result because SSD efficiency test can be seen as T-criteria problem while any mean-risk efficiency test is only two-criteria problem.

For future research, this study can be improved in various ways. For example, longer historical data can be used. In addition, one can consider the portfolio efficiency in a more robust way as it was done in [5] and [2] using, for example, contamination techniques discussed e.g. in [1]. Alternatively, one can apply fuzzy approach recently used in [3]. Unfortunately, all these improvements would lead to more computationally demanding efficiency tests what requires much better hardware equipment than is currently available.

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